Newton's Principia
M.Sci. Project

Benjamin Clifford,
Queen Mary and Westfield College

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Abstract

Newton's *Principia* provides the basis for modern Newtonian dynamics. Many of the results have survived the last three hundred years with little or no change.

In this report, an English translation of a part of the text is presented, along with commentary.

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Chapter 1

Background to the Translation

1.1 Introduction

Philosophiae Naturalis Principia Mathematica, Sir Isaac Newton's great work, provides the foundations for what is now known as Newtonian mechanics. Beginning with elementary definitions of concepts such as mass and momentum, the text rapidly builds to complex results on planetary motion.

I have translated parts of the first five parts of the text: The definitions, the laws of motion, section I on limits, section II on centripetal forces and the first result of section III on bodies moving in conic sections.

It is interesting to note the features present in the modern subject but missing in the original. For example, there is no concept of energy.

There are no cartesian co-ordinates. Corollary II shows that a force can be broken up or composed along any two directions. In the text, the directions taken are those convenient for the particular problem - for example, the direction that a force is acting, or that a body is travelling - rather than specific axes. [1] Consequently, there are no cartesian vectors for position, force, momentum or acceleration.

There are no explicit functions, although similar objects exist. For example, Lemma I refers to quantities that vary with time, and Proposition V refers to a velocity that varies at different positions around an ellipse. These "functions" are assumed to have nice & natural properties, reflecting reality. They are implicitly analytic.

In the problems of Sections II & III, Newton asks for laws that relate the centripetal force to the position. These laws are functions from position to force and are distinct from the axiomatic laws presented nearer the start of the text.

Newton does not use abstract numbers: they must have some physical manifestation, either as a physical quantity (for example, a length, an area or a solid) or as a ratio of physical quantities. These ratios correspond to the modern concept of real numbers, as they are ratios of arbitrary quantities, rather than ratios of integers.

[1] Descartes, for whom cartesian co-ordinates are named, lived 1596-1694, contemporary to Newton.

Quantities must always be positive (this follows naturally if you insist that quantites can actually be represented by something physical). Therefore, ratios of quantities must always represent positive reals. This is necessary for the proof of Lemma IV - if we permit negative quantities, then a counterexample is possible. He uses the word *number* (actually the latin word for number) in Lemma IV. This means natural number, rather than real number.

It is never explicitly stated that quantities should be positive, but it is important to consider the historical context here. In Newton's time, the concept of negative values existed but such numbers were second class. They were not accepted just anywhere where a normal (positive) value was permissible, and seen as really just a trick to make certain things easier. So, it would never have occurred to Newton to state that his quantities and ratios must be positive.

1.2 Translation

The full title of the text is, literally, *The Mathematical Principles of Natural Philosophy*. Here, in the title, we hit the first stumbling block of translation. The latin phrase *Philosophiae Naturalis* means *Natural Philosophy* when taken literally, but in modern English means something more like *Physics* or *Science*. Which should I choose?

In producing this translation, I have often been forced to pick one of several possible translations for words and phrases. There is a constant struggle between translating word-for-word and conveying the meaning incorrectly or cloudily, and translating what I believe to be the concepts behind what is being said, at the risk of mixing Newton's original work with more modern ideas and concepts.

One problem with translating specifically this work is that everyone is used to Newtonian dynamics - many of the results are "obvious" because of repeated exposure since an early age, even though they were not at the time.

I will now address some specific problems with translating from Latin to English.

Latin places fewer restrictions on word ordering than English. Some of Newton's sentences, in this translation, may appear strangely ordered. The language can also be very concise, using a sentence for what could easily translate into a full English paragraph. I have sometimes stuck closely to the original, but at other times I have rephrased sentences or paragraphs.

In some cases, literal translation of Latin would give "If a body would be rotating", for a phrase that would be more meaningful as "Let a body be rotating".

Various words are used to denote the operation of squaring, or raising to the second power, but these words often could be translated as *doubled*[2], or *twice*[3].

The Latin word *gravitas* translates to weight. Newton also uses it to mean gravity, as in *the force of gravity*, or *the force of weight*. A moment's contemplation reveals that these are exactly the same concepts: modern English has acquired two words for this one Latin word (although the English word *weight* is often used, in common speech, as if it meant *mass*).

Latin does not have words for *the* or *a*. They are usually implied, although sometimes replacements such as *one* are used if required. In Proposition X.,

[2] duplus
[3] bis

a/the law of centripetal force is required. In an English statement of the problem, if the word *the* was used, there would be an implication that the solution is unique, and if *a* was used, then it would be suggested that there may be several solutions. There is no such implication in the Latin.

We would usually say: "considering A,B,C we can label the forces A', B', C' *respectively*". Newton doesn't use an equivalent word. It is implied that if he lists two sets of things, they are paired up in order. The word "respectively" is there implicity.

This Latin word *vis* is used to mean *force*: for example, *vis centripeta* is centripetal force. It has a more general meaning than specifically that thing we now measure in units of N: for example, Definition III refers to *vis insita*, which means the intrinsic force or instrinsic strength, or mass.

The word for tangent is usually used as verb, meaning "touches".

1.3 Notation

In the original text, Newton often uses a familar notation; at other times he has other conventions.

$\times$ means multiplication (as now), of quantities or ratios.
$\frac{a}{b}$ means real division (as now).
A^n is used as now, but only very rarely (for example in Proposition IV).

GvP means $(Gv) \times (vP)$ where G, v and P are co-linear points and Gv and vP are the distances between those points.

I have written $A : B$ for the ratio of A to B. Newton does not use such notation, instead expressing the ratio as prose, *A ad B*, or sometimes as a fraction.

There is no symbol for proportionality in the original text. The Latin word *ut* is used in prose. I have often translated to symbolic notation.

Newton often swaps the order of points in his text. At one point, he will talk about AC, and then suddenly will switch to CA.

1.4 Dates

The dates in Newton's life in table 1.1 are from p393 of [1].

1.5 Contemporaries

- Sir Christopher Wren - mentioned in text (as Wrennus). Architect of St. Paul's Cathedral. He was a mathematician, then a physicist and then an architect. Lived 1632-1723. He was president of the Royal Society.

- Edmund Halley - also mentioned in text

- Christian Huygens - 1629 - 1695 - discovered cycloids.

- Leibniz - Newton/Leibniz disagreed about who invented Calculus. Lived 1646 - 1716. In the first edition of the *Principia*, Newton acknowledges that Leibniz had similar ideas, but these were removed in later additions,

Table 1.1: Dates

Year	Newton's Life	General History
1642	Born	
1649		Charles I beheaded
1660		Charles II crowned - Restoration of the monarchy.
1661	Goes to Trinity College, Cambridge	
1662		Royal Society founded
1665	Awarded A.B. degree, started to make own discoveries	
1665-1666	Trinity College closed due to plague. Newton went home and (it is claimed) discovered: the binomial theorem (for non-integer coefficients); calculus; the law of gravity; the nature of colours.	The Great Fire of London
1671	Introduced "pricked letter notation", $\dot{x}$ to represent the rate of change of x (the fluxion of the fluent x).	
1687	1st Edition of *Principia* (London), the first printed account of calculus.	
1713	2nd Edition of *Principia* (Cambridge)	
1726	3rd Edition of *Principia* (London)	
1727	died [4]	

after they had fallen out. Leibniz invented the notation that we use today: the big S, $\int$ for integrals and dy and dx for differentials.

Chapter 2

Definitions

In these definitions, Newton introduces mass, momentum, general forces and centripetal forces.

He introduces mass twice: firstly in Definition I as a product of density and volume, and then secondly in Definition III as a measure of how a body is to move. He does not, however, define density. Nor does he define velocity, which is used in the definition of momentum (Def. II).

The later definitions deal with centripetal forces. In the modern teaching of mechanics, centripetal forces are often neglected (with linear systems preferred), but they play a key role in the *Principia*.

Quantities such as momentum are defined proportional to other quantities. In modern texts, we would usually define them with an equality relation rather than a proportionality relation, by absorbing the constant of proportionality into the units used: For example, mass is equal to the density times the volume if we pick a suitable system of units such kilograms, kilograms per cubic metre, and metres.

Definition I. *The amount of material*[1] *is a measure arising from the density and size combined.*

[2]

Air, doubled in density and also doubled in volume is quadrupled. When tripled, it is increased sixfold. By the same, snow and powder, when compressed or melted, are made denser.

Definition II. *The amount of motion is a measure arising from the velocity and amount of material combined.*[3]

---

[1] mass

[2] Size means volume.

$$M \propto d \times V$$

where M is mass, d is density and V is volume. In modern times, we would usually write this definition as a "division" : $d = \frac{M}{V}$

[3] We would use the Latin-sounding word "momentum" for this quantity – Newton uses "quantitas motus". The amount of motion is the state of the body, as used in the following definition and later.

$$p \propto v \times M$$

where p is momentum, v is velocity, M is mass.
It is implicit here that the body is rigid and non-rotating - in modern times, we would often

Total motion is the sum of motion in single parts; and by the same reason, with doubled body, with equal velocity, (the motion) is doubled; and with doubled velocity, it is quadrupled.

Definition III. *The intrinsic strength of a material is its potential to resist, by which means every body, according to its extent*[4]*, remains in its state, either stationary or moving uniformly in a direction.* [5]

Definition IV. *Impressed force is action on a body to change its state - either remaining still or moving uniformly in one direction.*

Definition V. *Centripetal force is the means by which bodies are drawn, pushed or in any way tend towards a central point.* [6]

An example of this type of force is gravity[7] by which bodies are drawn to the centre of the earth;

magnetic force, from which iron seeks a magnet;

and that force, whatever it is, from which the planets are perpetually drawn from motion in a straight line and are forced[8] to revolve in a curve.

A stone, in a spinning sling, attempts to move away from the driving hand, and from its attempt, the sling stretches and the faster it goes, the stronger it is forced out, so that the stone is thereby accelerated[9] and revolved; and as soon as it is released, the stone flies away.

The quantity of this centripetal force is of three types: absolute, accelerating and motive[10].

Definition VI. *The absolute amount of a centripetal force is proportional to the effect it causes towards the centre from regions in the circuit.*[11]

Magnetic force from a magnetic stone[12] by virtue of intensity[13] is larger in one magnet, smaller in another.

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talk of many of these definitions applying to a particle (although not Def. I as a particle has no size).

It has been suggested that what is actually meant here is that the total momentum of a body is an integral of over the volume of the body. I disagree. Corollary III deals with the case of combining bodies together into a system and I believe it applies equally well when those bodies are fixed together (i.e. parts of a composite body).

[4] the strength arising from the very existence of the material rather than any external thing

[5] This is a second definition of mass, as an intrinsic strength of material; mass is the amount by which the object resists change in its momentum. We have two definitions of mass, one called "vis insita" and one called "quantitas materiae". Newton comments after this definition that these are equivalent, except for the way of conceptualising them.

[6] I have always seen centripetal force used in, for example, a circle. However, Newton uses centripetal force in a more general sense - a centripetal force is any force which deflects a body from its straight path towards a specified point (which may or may not be moving), and therefore goes naturally alongside the "vis insita" (intrinsic strength / intrinsic force) in Def. III. Vis insita is the thing which makes the body go straight, and vis centripetae is the thing which moves it off this straight.

[7] The Latin word used here is *gravitas*. This can be translated to the modern word gravity, but it could also mean weight, or heaviness – see the Introduction.

[8] The word here is *cogo = forced* rather than *cogito = known*.

[9] accelerated inwards

[10] to do with motion or momentum

[11] The absolute amount is the magnitude.

[12] a magnet or lodestone

[13] the magnet's magnetic strength

Definition VII. *The amount of acceleration of a centripetal force is itself a measure proportional to the velocity which it generates in a given time.*[14]

The force from a magnet is greater at a smaller distance, and smaller at greater distance.

Gravitational force is larger in a valley and smaller on the peak of a high mountain,[15] and even smaller still (as will be evident in the future) at a greater distance from the globe of the Earth. However, in all directions, all bodies (heavier or lighter, greater or smaller) at equal distance feel the same acceleration (neglecting air resistance).

Definition VIII. *The amount of motion of a centripetal force is a measure proportional to the momentum which it generates in a given time.* [16]

[14] $a \propto v_{t=1} - v_{t=0}$. This is true for any force, though, in short periods of time. I think the given time here is implicity short.

[15] This is true for a mountain, but not necessarily so in a valley. Descending below the surface of the earth, gravitational force reduces to 0 at the centre

[16] This is an analogue of def 7, but for momentum rather than velocity.

Chapter 3

Axioms or Laws of Motion

This section presents the famous three laws of motion.

The first and third laws have remained basically intact, but the second law has changed considerably over time. A footnote at the appropriate point discusses the changes.

The corollaries which follow the laws provide results on splitting and combining forces into components, and on various properties of multi-body systems.

Law I. *Every body will remain as it is, either stationary or moving uniformly in a direction, except for how much it is forced to change its state by impressed forces.*[1]

Projectiles continue in their motion, except to the extent that they would be retarded by air resistance and impelled downwards by the force of weight[2].

A hoop, of which the joined parts perpetually drag themselves from rectilinear motion,[3] does not cease to rotate, except for how much air resists.

Indeed, the larger bodies of planets and comets moving and progressing and circulating in space, with smaller factors of resistance, conserve their motion for longer.[4]

Law II. *The change of motion is proportional to the impressed force of motion, and happens according to a straight line that the force acts.* [5]

If any force generates some motion; doubling will generate double, tripling

[1] This is conservation of momentum.

[2] or gravity

[3] When a hoop is spinning, each part of the hoop cannot fly off at a tangent because it is joined to the rest of the hoop.

[4] There is a point made here about the nature of the universe. Newton is saying that we can deal with small human-scale objects using exactly the same laws as we can use to deal with celestial bodies. In Newton's time, there was a lot of argument about the nature of the universe.

[5] The change in momentum happens in the direction of the force, and is proportional to that force.
Law II is commonly quoted nowadays as $F = ma$. This appears to be quite different to the Second Law quoted above, which relates change in momentum to force. However, by applying Definition II, we can convert this change in momentum into change in velocity, or acceleration. The original law also comments on the direction in which the change takes place. This is absorbed, in the modern statement, into vector notation, with F and a as vectors and m a scalar.

will generate triple, whether together and at once or by degrees and successively.[6]

And, if a body was previously being moved, this motion (because it is always restricted to the plane of the generated force) itself either, acting in the same direction, adds, or acting in the opposite direction, takes away, or acting obliquely, combines obliquely.

Law III. *To every action is always an equal an opposite reaction: if two bodies act upon each other, the forces are equal in magnitude and in opposite directions.*

Whatever presses or draws another body, just as much is pressed or drawn by that body. If anyone presses on a stone with a finger, his finger is pressed by the stone.

Corollary I. *In a given time, a body will describe, by two combined forces, the diagonal of a parallelogram, the sides of which it would describe from the component forces in the same time.* [7]

Figure 3.1: Parallelogram

If a body, in given time, a single force M at position A having been impressed, would be borne with uniform motion from A to B; and a single force N at the same position[8] having been been impressed, would be borne with uniform motion from A to C: then let the parallelogram $ABDC$ be completed, and by the resulting force, that body shall be borne from A to D.

For, because force N acts on the line AC, itself parallel to BD, this force by Law II, does not change the velocity generated by the other force towards that line BD.

Therefore, that body approaches the line BD in the same time, whether force N is impressed or not;

And so at the end of that time, it is found somewhere on that line BD.

By the same argument, at the end of the same time, it is found somewhere on the line CD.

And therefore, because the resulting lines meet at D, it moves in a straight line from A to D, by Law I.

[6] If we double the force, we will change the momentum by twice as much, and if we triple the force, we will change it by three times as much, It doesn't matter how the total force is applied - either as a single thrust, or as several separate successive thrusts, it will cause the same change in momentum.

[7] This is the parallelogram law.

[8] A

Corollary II. *And hence is evident, the composition of the arbitrary force of direction AD from arbitrary oblique forces AB and BD, and conversely the resolution of an arbitrary force in direction AD into arbitrary oblique forces AB and BD. Indeed, this composition and resolution is abundently confirmed from mechanics.* [9]

Corollary III. *The amount of motion which is collected taking the total motions of parts to the same direction and subtracting factors in opposite directions, is not changed by the action of the bodies on each other.*[10]

Corollary IV. *The common centre of gravity*[11] *of two or more bodies, as a result of the actions of the bodies on each other, does not change its state, either moving or stationary, and therefore, as a result of their mutual interactions (excluding external actions and impediments) the common centre of gravity of all the bodies either is stationary or moves uniformly in a direction.*[12]

Corollary V. *The relative momentum of bodies included in a space is the same whether that space is stationary or moving without circular motion.*[13]

For the differences of the motions tending in the same direction, and the sums of the motions tending in the opposite direction, are the same in the former and latter case[14], by hypothesis. And from these sums or differences arise the total forces by which the bodies are mutually borne. Therefore, by Law II, the effects on each other would be equal in both cases. And, therefore, momentum relative to each other in one case would be equal to momentum relative to each other in the other case.

Corollary VI. *If bodies would be moved somehow amongst each other, and would be urged from an equally accelerating force acting along parallel lines, they will move amongst themselves as if they were not affected by that force.* [15]

[9] In vector notation, $AD = AB + BD$. We can combine two forces into one and resolve one into two.

[10] If we have a system of bodies acting on each other, the total momentum of this system is not changed by the forces of the bodies on each other.
It is necessary here for Newton to explicity mention adding motions in the same direction and subtracting motions in opposite directions because of the lack of negative numbers.

[11] or centre of weight. Centre of mass would be an (almost?) equivalent concept that modern readers will be more familar with.

[12] The common centre of gravity of a system is not changed by the action of forces amongst the bodies in a system.

[13] A "frame of reference" has uniform motion.

[14] The former case is the stationary case, the latter is the moving case.

[15] If we have a "frame of reference" in which everything is being accelerated the same amount, the bodies will act amongst each other as if that acceleration wasn't happening - we can't tell if there is external acceleration just by looking at the bodies.

Chapter 4

Section I. Ultimate Ratios

This chapter presents a theory of limits. The concept of the *ultimate* or *final ratio* is introduced, this being the limiting case of a ratio of two quantities.

The first lemma considers the case where two quantities can be made closer together than any given difference. The following lemmas deal with areas under curves, by establishing upper and lower bounds and taking the limit of those bounds as they are made increasingly more accurate, and with other properties of curves that can be deduced by limits.

This section is actually given the somewhat less snappy title: *On the method of first and ultimate ratios, with the aid of which consequences will be demonstrated.*

Lemma I. *Quantities, and also ratios of quantities, which tend*[1] *in an arbitrary finite time to a constant and before the final time tend nearer to each other that any given difference, will become ultimately equal.*

If not, then ultimately they will be inequal. Let their ultimate difference be D. Therefore, they are unable to approach nearer to equality than difference D. Contradiction. [2]

Lemma II. *If in an arbitrary figure $AacE$, bounded by straight lines Aa & AE and curve acE would be inscribed the parallelograms*[3] *Ab, Bc, Cd and so on, with equal bases AB, BC, CD, and so on, and lateral sides Bb, Cc, Dd parallel to the side Aa, and the parallelograms $aKbl$, $bLcm$, $cMdn$ and so on would be completed, then as the width of these parallelograms is made smaller and the number increased to infinity: I say, the final ratios of the inscribed figure $AKblcMdD$, the circumscribed figure $AalbmcndoE$ and the curve $AabcdE$ are the ratios of equality.*[4]

[1] It is necessary here for this to be monotonic, although Newton does not comment upon it. This property follows from his assumption that all functions are analytic - over a suitably small range, they are always monotonic.

[2] This is reminiscent to the "For all epsilon, there exists a delta" definition of convergence.

[3] In the diagram, these are rectangles, but it does not appear that this is necessary for the lemma to work.

[4] The ratio of equality is simply the ratio 1:1. To say several quantites are in the ratio of equality is to say that they are equal. So this lemma says that the areas of the inscribed, circumscribed and curved figures are ultimately equal: they converge.

Figure 4.1: Rectangles inscribed and circumscribed on a curve

For, the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl, Lm, Mn, Do, this is (because every base is equal) a rectangle of a single base Kb and total height Aa, i.e. the rectangle $ABla$. But this rectangle, because the lattitude is made infinitely smaller, would be smaller than any given. Therefore, by Lemma I, the inscribed and circumscribed, and the many intermediate curved figures would become ultimately equal. Q.E.D.[5]

Lemma III. *The same final ratios are also ratios of equality when the bases of the parallelograms AB, BC, CD are inequal and made infinitely small.*[6]

Indeed, let AF be equal to the maximum side[7], and let the parallelogram $FAaf$ be completed. This will be bigger than the difference of the inscribed and circumscribed figures. But then the side AF having been infinitely diminished, is made smaller than any given rectangle. Q.E.D.

Corollary 1 Hence, the ultimate total of the vanishing parallelograms coincide in every part with the curved figure.[8]

Corollary 2 And even more so, the rectilinear figure whose chords vanish[9] into the bounding arcs ab, bc, cd, coinciding ultimately with the curved figure.[10]

Corollary 3 And the same for a circumscribed figure with boundaries that are tangents of the arc.

Corollary 4 And therefore, these ultimate figures (with perimeter acE) are not straight-sided, but curved limits of straight-sided figures.

[5]This proof actually shows that any figure trapped between the inscribed and circumscribed figures will converge.

[6]The figures in Lemma III still converge, even if the bases of the parallelograms are unequal.

[7]Latitude in the text, Latin: *latitudine*

[8]Lemma III says that the areas of the various shapes are the same. This corollary says that the shapes are congruent.

[9]The Latin phrase used here is *evanescentium*, meaning *vanishing*, but the meaning here is more like *converging to* or *merging with.*

[10]If we replace the rectangles with trapezia with horizontal bases and chords at the top, we still get convergence, and that convergence is even faster.

Lemma IV. *If in two figures $AacE$, $PprT$ are inscribed (as above) two series of parallelograms, and the number would be the same, and when the bases are diminished infinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other (one to one) would be the same: I say that the two figures $AacE$, $PprT$ are to each other in the same ratio.*

Figure 4.2: Two Curves with Inscribed Rectangles

For the ratio of the parallelograms (one to one) is (by componendo[11]) the ratio of the sum of all the parallelograms in one figure to the sum of all the parallelograms to the other, and the ratio of one whole figure to the other.

Evidently, the first figure is equal to the sum of the first set of parallelograms, and the second figure is equal to the sum of the second set of parallelograms (by Lemma III).

Q.E.D.

Corollary Therefore, if two quantities (of any kind) are divided into the same number of parts, and these parts, when the number is made infinitely larger and the size indefinitely smaller, tend to a given ratio to each other, first to first, second to second and others to others in order, then the totals[12] will be in the same given ratio to each other. [13]

proof of corollary For if in the figures in the lemma, the parallelograms (paired with the parts)[14] are summed, the sum of the parts will always be proportional to the sum of the parallelograms. And also when the number of parts and parallograms increases infinitely and sizes decrease indefinitely, the final ratio of parallelogram to parallelogram is (by hypothesis[15]) the final ratio of parts to parts.

[11] See Useful Result 3

[12] the original quantities

[13] This corollary generalises Lemma IV to any quantities, not just areas as in the lemma. It is necessary for quantities to be positive here - see the discussion on this in the introduction to this report.

[14] The parallelograms are to be assigned to the parts in such a way that the areas of the parallelograms in the lemma are proportional to the corresponding parts in the corollary.

[15] By the hypothesis of the Lemma, not of the corollary

Lemma V. *Corresponding sides of similar shapes are proportional to each other, as much for curved sides as for straight sides; and the ratio of the areas is the square of the ratio of the sides.* [16]

Lemma VI. *To any arc ACB, let there be drawn a chord AB and through any point A, in the middle of continuous curvature*[17]*, let there be produced*[18] *a tangent AD, and then let points A, B approach and merge together; I say that $\angle BAD$, between the chord and the tangent, would be made infinitely smaller, and finally vanish.*

Figure 4.3: Curve with various lines

For, if that angle does not vanish, arc ACB and tangent AD will enclose a rectilinear angle, and therefore the curvature to point A will not be finite, contrary to hypothesis.[19]

Lemma VII. *With the same assumptions: I say the ultimate ratios of the arc, the chord and the tangent to each other are the ratio of equality.*[20]

For, while point B approaches point A, let AB and AD be produced to distant points b and d, with bd drawn parallel to BD and let arc Acb always be similar to the arc ACB.

As points A, B come together, $\angle dAb$ vanishes by the above lemma.[21] And similarly the straight finite lines Ab, Ad and the intermediate arc Acb always coincide and therefore will be equal. From which, the proportional straight lines AB and AD and intermediate arc ACB always vanish[22], and will have the ultimate ratio of equality.

[16] It is quite hard to distinguish this from the actual definition of similarity. Two sides are similar if one can be uniformly stretched and then moved so that it is congruent to the other.

[17] What is meant by continuous curvature? That there are no "sharp" angles - what we would now regard as discontinuities in the gradient of the curve. The gradient is never 0 or infinite and is always positive (of course - everything is positive in Newton's world!).

[18] Producing a line means extending it.

[19] If the curve is continuous, the angle of the arc and the tangent should approach zero when A and B are close. A rectilinear angle is an angle with straight sides.

[20] The arc, chord and tangent are equal in the limiting case.

[21] Lemma VI

[22] vanish into each other

Q.E.D. [23]

corollary 1 And so, if from B would be drawn BF, parallel to the tangent, with whatever straight line AF from A always crossing at F, this BF ultimately will have the ratio of equality to the vanishing arc ACB, for the reason that completing the parallelogram $AFBD$, it always has the ratio of equality to AD.

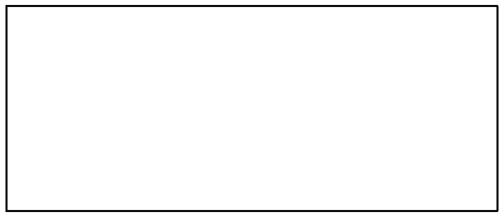

Figure 4.4: Variant of previous figure

corollary 2 And if from B & A are drawn several straight lines BE, BD, AF, AG, cutting tangent AD and its parallel BF; the ultimate ratio of every abscissa AD, AE, BF, BG, chords and the arc AB to each other will be the ratio of equality.

corollary 3 And therefore, all of these above mentioned lines in arguments about ultimate ratios are able to take the place of each other. [24]

Lemma VIII. *Given straight lines AR, BR with arc ACB, chord AB and tangent AD, let the triangles RAB, $RACB$, RAD be formed, and let the points A, B be brought together: I say the final forms of the vanishing triangles are similar and ultimately are equal.*

For while point B moves to point A, lines AB, AD, AR would produce distant points b, d and r, where rbd is drawn parallel to RD, and arc Acb is always similar to ACB.

Bringing together points A and B, $\angle bAd$ vanishes and therefore the three finite triangles $\triangle rAb$, $\triangle rAc$, $\triangle rAD$ cooincide and are similar and equal.

And so, these similar and proportional $\triangle RAB$, $\triangle RACB$, $\triangle RAD$ become finally similar and equal to each other. Q.E.D.

Corollary And these triangles in every argument about limits, can be substituted for each other.[25]

Lemma IX. *If straight line AE and curve ABC in given position mutually intersect each other at a given angle at A, and to that straight line in an given angle would be applied perpendiculars BD, CE meeting the curve at B, C, then as points B, C simultaneously approach point A: I say that the areas of the triangles $\triangle ABD$, $\triangle ACE$ will be finally to each other in the square of the ratio of the sides.*

[23] It is not entirely clear to me why Newton has to construct this larger similar shape to argue about - why do the arguments not apply to the smaller shape itself? Is he just trying to make the proof more complicated? Am I missing something? In the variant reading, his proof of the previous lemma, Lemma VI, also uses this extended lines so maybe this proof is a natural follow-on from that previous proof.

[24] When dealing with ultimate ratios, we are able to exchange any ratio of the above lines with any other ratio.

[25] This corollary is the equivalent of Corollary 3 of Lemma VII for triangles rather than lines.

Figure 4.5: Curves and triangles

For, while points B, C move to point A, the line AD will always be understood to be produced to distant points d, e, such that Ad, Ae are proportional to AD, AE and would form ordinates db, ec parallel to DB, EC which would meet the extensions of AB, AC at b and c. It would be understood to be drawn as much as curve Abc itself similar to ABC, as right line Ag, which is tangent to the curve at A. and it would cut the applied ordinates DB, EC, db, ec at F, G, f, g. As longitude AE would remain, B and C would come together to point A; and angle $\angle cAg$ vanishes, areas of curvilinears Abd, Ace coincide with rectilinears Afd, Agc; And by the same reasoning (and lemma V) they will be in square of the ratio of the lines Ad, Ae: But these areas are always proportional to areas ABD, ACE, and these lines to AD, AE. Therefore areas ABD, ACE are ultimately in the square of the ratio of lines AD, AE.

Q.E.D.

Lemma X. *Spaces*[26] *described by bodies which are urged by some finite force, that force is either determined and fixed, or continuously increasing or decreasing, would be at the beginning of the motion in the square of the ratio of times.*[27]

Let time be represented[28] by the lines AD, AE, and generated velocities by ordinates DB, EC, and the spaces that these velocities describe will be proportional to areas ABD, ACE that the ordinates describe, that is, (by Lemma IX.) at the beginning of the motion, they will be proportional to the square of the times AD, AE.

Scholium If separate variable quantities would be joined together[29] with each other, and one is proportional to another, either directly or inversely: then this

[26] lengths or distances

[27] $s \propto t^2$ at the start of the motion of two stationary bodies under some force that is continuously increasing or decreasing. If the force, and hence acceleration, were always fixed then this relationship would hold in the non-limiting case as well.

[28] Latin: *exponantur* which means set-out or explained. I think represented is a better word here.

[29] by a proportion

means that the former is diminished in the same ratio[30] as the latter, or with the reciprocal of the same.

And if the same is said to be proportional to two or more others, either directly or inversely: then, it is meant that the product is added or diminished in a ratio which is made from the ratios in which the others or reciprocals of the others are increased or diminished.

If A is directly proportional to B, directly to C and inversely to D: then A is increased or decreased in the same ratio[31] as $B \times C \times \frac{1}{D}$. That is, A & $\frac{BC}{D}$ are to each other in the given ratio. [32]

Lemma XI. *A subtangent[33] across a vanishing angle of contact, in every curve with finite curvature at that point, is ultimately proportional to the square of the arc or the chord.*[34]

Figure 4.6: Curve with lines to centres of curvature

Proof (in 3 cases)

Case 1 Let that arc be AB, the tangent to it AD, subtangent BD, subtense of the arc AB. [35]

To this subtense AB and tangent AD, perpendicularly would be drawn AG, BG meeting at G.

Then if the points D, B, G would move to the points d, b, q, and I would be the intersection of the lines BG, AG ultimately when the points D, B move to A.

[30] at the same rate

[31] or *increased or decreased at the same rate*

[32] In this scholium, Newton discusses the relationship of ratios and proportions. He says: $(A \propto B, A \propto C, A \propto D^{-1}) \Rightarrow A \propto \frac{BC}{D}$.

[33] BD on the diagram

[34] Newton says that it is ultimately proportional to the *subtensa* which, at least in the limiting case, can be either the length of the arc or the length of the chord.

[35] The above mentioned angle of contact is the point A.

It is evident that the distance GI can be smaller than any assigned distance.[36]

However (from the nature of circles through points ABG, Abg) $AB^2 = AG \times BD$; $Ab^2 = Ag \times Bd$ and so the ratio $AB^2 : Ab^2 = AG : Ag \times BD : bd$ [37]

But since GI can be assumed to be smaller than any assigned distance, then the ratio $AG : Ag$ can be made to differ from the ratio of equality less than any assigned difference.

Therefore, by Lemma I, the ultimate ratio $AB^2 : Ab^2$ is the same with the ultimate ratio of $BD : bd$.

QED.[38]

case 2

Now, let BD be inclined to AD at any given angle, and the ultimate ratio $BD : bd$ will always be $AB^2 : Ab^2$, by the same.

Q.E.D.

case 3

And if angle D is not given, but line BD converges to a given point or another law is constituted: then nevertheless, common angles D,d always head towards equality, and therefore get closer to each other than any given difference and so will be ultimatly equal by Lem. I, and therefore lines BD, bd would be in the same ratio as the first case.

Q.E.D.

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[36] I is the centre of curvature of the point A.

[37] $\triangle ABD$, $\triangle GAB$ are similar, so $AB/AG = BD/AB$.

[38] This case is suffices to prove the lemma. The other two cases extend the result to something more general.

Chapter 5

Section II. On the finding of centripetal forces

This section, containing Newton's first ten propositions, deals with centripetal forces in a variety of figures. They are the first results in the book which carry substantial proofs.

A number of his propositions are also labelled as problems, and are laid out as such, with the answer at the end of the workings. I have often "cheated" and reproduced the answers immediately after the statement of the problems.

Proposition I. Theorem I. *The area described by the radius from a body to a fixed centre, that body acting under a force to that centre, is proportional to time.* [1]

[needs FIGURE39CURVE]

Let time be divided into equal intervals.

In the first interval, let the body describe the straight line AB (from its inertia).

In the second interval, if nothing interfered with it the body would describe the line Bc (by Law I), with $Bc = AB$

Consider, the radiuses AS, BS, cS acting to the centre, making $\triangle ASB$ and $\triangle BSc$ of equal area.[2]

When the body goes to B, centripetal force pulls the body so rather than travelling along the line Bc, instead it travels along the line BC.

Let the line cC be drawn parallel to BS, intersecting BC at C.

and completing the second part of time, the body (by Corollary I of the Laws) will be found at C, in the same plane as $\triangle ASB$.

Joining S to C and because $SB \parallel Cc$, $\triangle SBC = \triangle SBc = \triangle SAB$. [3]

[1] The body can be moving in any curve - it is not limited to ellipses or circle. The curve doesn't even need to be a closed loop. It can be any curve with suitable properties, described in a previous footnote.

[2] equal area because: base line Ac, triangles have the same length on the bases and are the same height (the height of S down to the base line). two triangles with the same base width and same height have equal area. see Useful Result 1

[3] consider the baseline SB. Both of these triangles have this base. The only difference between the two triangles is that their apex has been moved along the line Cc, parallel to the baseline. So, they have the same height. Same height and same base $\Rightarrow$ same area. So first equality holds. We have already shown the second equality, above.

Figure 5.1: Centripetal Forces

The same argument applies so that if the centripetal force is successively applied at C, D, E, etc., the body in a single interval describes the straight lines CD, DE, EF, etc.

These all lie in the same plane, and $\triangle SCD = \triangle SBC = \triangle SDE = \triangle SEF$.

Therefore, equal areas are described in equal times, and putting together the total areas, $SADS : SAFS = t_{AD} : t_{AF}$ [4]

Increase the number of triangles and decrease their width infinitely.

The final perimeter ADF is a curved line (by corollary 4 of Lem. 3)

By this idea, the centripetal force, which perpetually pulls on the body from its tangent, will always pull indefinitely.

Indeed, the areas which are described, $SADS$, $SAFS$ are always proportional to time.

Q.E.D.

Corollary 1 The velocity of a body attracted to an immobile centre in non-resisting space, is reciprocally proportional to a line to that centre, perpendicular to the tangent of the orbit.

Indeed, velocities in those positions A, B, C, D, E, are proportional to the bases of the equal[5] triangles AB, BC, CD, DE, EF, and these bases are reciprocally proportional to the perpendiculars dropped from them.

[4] My notation t_{PQ} means the time taken to travel from P to Q.

[5] equal in areas

Corollary 2

When these arcs are infinitely diminished, the line BV will ultimately cross the centre S.

corollary 3 If two arcs of equal time in non-resisting space, describe chords AB, BC, and DE, EF, let the parallelograms $ABCV$ and $DEFZ$ be completed. The ratio of the centripetal forces at B and E is equal to the ratio of the diagonals of the parallelogram,[6] when the arcs are infinitely diminished.

corollary 4 The forces by which some arbitrary bodies in non-resisting space are pulled or turned in curved orbits would be proportional to the arrows of the arcs of equal time which converge to the centre of the forces and bisect the chords, when these arcs are infinitely diminished.[7]

proof For these arrows would be as half the diagonals mentioned in corollary 3.

Proposition II. Theorem II. *Every body, which is moved in some curved line, in a described plane* [8]*, where a radius drawn to a point either immobile or proceeding with uniform straight motion describes areas around that point proportional to time, would be urged by a centripetal force to that same point.* [9]

Case 1 (for fixed centre) Every body which is moved in a curved line, is turned away from the straight path by some force (by Law I). Considering the small equal[10] triangles SAB, SBC, SCD, and so on around an immobile point S described in equal times, the force by which the body is turned away descibes, at position B, a straight line parallel to cC (by Law II, and I.40 of Euclid), which is according to the straight line BS.

And, at position C, a straight line itself parallel to dD, this is the line SC, and so on. Therefore, it always describes lines tending to that immobile point S. QED.

Case 2 (General case) By Corollary V, the space containing the bodies, the described figure and the point S, either moves uniformly in direction, or rests.

Corollary In a non-resisting space or medium, if areas are not proportional to time, then forces do not tend to the intersection of the radial lines. Instead, the forces deviate in advance of the line in which it moves, if the areas are increasing, and if the areas are decreasing, the forces deviate against the line.[11]

[6]
$$F_B : F_E = BV : EZ$$

[7][supplemental diagram from me: Parallelogram ABCV, with line AC labelled Chord and dotted line from C to V, with a thick arrow from C to the point in the middle where the lines cross] The arrow appears to be as drawn in this diagram. I am not sure where it is defined (or if it is defined at all?) It is a line that is the deflection produced by the force - if the force did not exist, the body would proceed because of its inertia to a different point, and the arrow of the force is the distance that we then need to move the body to where it really ends up because of the force. We can decompose the movement over a time into two components, the inertial component and the arrow which is the distance produced by the force.

[8]a fixed plane

[9]This is the converse of Proposition I.

[10]in areas

[11]If the areas described are increasing with time, the force has a component in the direction

Proposition III. Theorem III. *If a radius of a body to the centre of another body which is moving somehow, describes areas around that centre proportional to time, the body would be urged by a composite force of a centripetal force to the other body and a force which accelerates in the same way that the other body is accelerated.*[12]

Let the first body be L, the other T: and (by Corollary 6 of the Laws) if a new force which would be equal and opposite to that which urges the other body T were applied to the bodies, they would be urged on some parallel lines; the first body L will advance describing areas around the other body T, areas the same as before: However, the force which the other body T was being urged will now be eliminated by force equal and opposite to itself, and therefore (by Law I), that other body T will now remain stationary or will be moved uniformly in a direction: and the first body L urged by the difference of the forces, that is, urged by the remaining force, describes areas proportional to time around the other body T.

Therefore (by Theorem II), it tends under the difference of forces to the other body T.

Q.E.D.

corollary 1 And so, if the body L by a radius drawn to another body T, describes areas proportional to time, and also, total force (either simple or composed of several forces as in Corollary 2 of the Laws) by which the first body L would be urged, is subtracted (by the same corollary) the total accelerating force by which the other body would be urged: every remaining force, by which the first body would be urged, tends to the other body, as a center.[13].

corollary 2 And, if those areas are approximately proportional to time, the remaining force tends approximately to the other body T.

corollary 3 And, vice versa, if the remaining force tends approximately to the other body T, then the areas will be approximately proportional to time.

corollary 4 If a radius of the body L drawn to the other body T describes which with times are exceedingly inequal, and that other body T is either stationary or moving uniformly in direction: the action of centripetal force to that other body T is either null or mixed and composed of actions of exceedingly different forces. And the total force of every force, if there are several forces, composed is steered to another (mobile or immobile) centre. By the same is obtained: when the motion of the other body is however moved, if centripetal forces would be summed, which remain after taking away the total force in the other body T.[14]

Proposition IV. Theorem IV. *Consider two bodies with equal momentum, which describe different circles,*[15] *and the centripetal forces tend to the centres*

---

that the body is moving. If the areas are decreasing, then instead there is a component acting against the direction that the body is moving.

[12]If both bodies are moving, and one body describes equal areas in equal time around the other, then the force on the first body is resolvable into a centripetal force, and a force which must accelerate the first body as much as the second body is accelerated. - so this second component will be a different force to that accelerating the second body if their masses differ.

[13]We can take away the component of the total force which accelerates both bodies, and what we are left with is purely centripetal force

[14]This last sentence doesn't make sense to me...

[15]In the diagram, they appear to have the same centre, but this is not necessary.

of those same circles; Then: the ratio of the forces is the ratio of the squares of the arcs simultaneously described, to the ratio of the radiuses of the circles. [16]

Proof[17]

[FIG44VariantConcCircles]

Let bodies B, b rotating in the circumferences of the circles BD, bd at the same time, describe arcs BD, bd.

Since under intrinsic force[18] on their own, they would be describing tangents BC, bc, these arcs equal in length, it is obvious that centripetal forces are what perpetually pull the bodies from the tangents to the circumferences of the circles, and these are to each other in the first ratio of the lines CD, cd. [19].

These centripetal forces indeed tend to the centre of the circles by Theorem II.

Because, the areas described by the radius are proportional to time.

If figure tkb is made similar to figure DCB, by Lemma V, the ratio of CD to kt will be the ratio of arc BD to arc bt.[20]

And so, by Lemma XI, the ratio of the lines $kt : cd$ will be $bt^2 : bd^2$.

And the ratio of the line DC to the line dc is equal to the ratio of $BD \times bt$ to bd^2.[21]

And therefore, the centripetal forces are proportional to $BD \times bt : bd^2$. And so, (because of the equality of the ratios $\frac{bt}{Sb}$ & $\frac{BD}{SB}$) $\frac{BD \times bt}{Sb}$ to $\frac{bd^2}{Sb} = \frac{BD^2}{SB}$ to $\frac{bd^2}{Sb}$. Q.E.D.

Corollaries [There are several corollaries which expess basic proportionality relationships. I have translated them into mathematical notation, rather than keeping them as prose.
R = radius
v = velocity
l = length of arc described in period time t
F = centripetal force.]

Corollary 1

$$F \propto \frac{v^2}{R}$$

Corollary 2

$$t \propto \frac{R}{v}$$

$$F \propto \frac{R}{t^2}$$

Corollary 3

---

[16] This proposition says: $F_B : F_b = \frac{BD^2}{SB} : \frac{bd^2}{Sb} = \frac{BD^2}{bd^2} : \frac{SB}{Sb}$ or that $F_B \propto \frac{BD^2}{SB}$.

[17] This proof has come from an earlier edition of the Principia - I find it easier to understand than the later version.

[18] inertia

[19] $F_b : F_B = cd : CD$

[20] $CD : kt = BD : bt$, the latter two being arc lengths rather than straight lines.

[21] By the similarity of the two figures above. $DC : dc = BD \times bt : bd^2$

If the period t is fixed, then:

$$v \propto R$$

$$F \propto R$$

and conversely.

Corollary 4

And, if p, v both $\propto R^{1/2}$ then F will be constant.
And contrary.

Corollary 5

If $t \propto R$ (and hence v is constant), then:

$$F \propto \frac{1}{R}$$

and conversely.

Corollary 6

If $t \propto R^{\frac{3}{2}}$, and therefore $v \propto R^{-\frac{1}{2}}$ then:

$$F \propto \frac{1}{R^2}$$

and conversely.

Corollary 7

If $t \propto R^n$ (and hence, $v \propto \frac{1}{R^{n-1}}$), then:

$$F \propto \frac{1}{R^{2n-1}}$$

and conversely.

Corollary 8

All the same results on times, velocities and forces, by which similar bodies describe any similar figures, have a centre positioned similarly in the figures, are demonstrated from the preceeding results applied to this case. However, equal described area must be substitued for equal motion, and the distance of the bodies from the centre must be substituted for the radiuses.

Corollary 9

From the same demonstration, it follows that the arc that a body describes in an arbitrary interval while the body is revolving uniformly in a circle under a given centripetal force is proportional to the square root of the product of the diameter of the circle and the descent of the body that occurs under the same given force over the same interval.

Proposition V. Problem I. *Given the velocities at points on a figure, where bodies in the figure act under forces to some common centre, that centre*[22] *is to be found.*[23]

Figure 5.2: Finding the centre of centripetal force

To the described figure can be made three tangents PT, TQV, VR at points P, Q, R, intersecting at T and V.

To the tangents, let there be errected perpendiculars PA, QB, RC made in reciprocal proportion to the velocities at those points P, Q, R.

Therefore, PA to QB would be as the velocity at Q to the velocity at P, and QB to RC would be as the velocity at R to the velocity at Q. [24]

From the ends A, B and C of the perpendiculars, at right angles, let AD, DBE, EC be drawn, intersecting at D and E.

And the lines TD, VE intersect in the found centre S.

For the perpendiculars hanging from tangents PT, QT to the centre S (by Proposition I corollary I) are reciprocal to the velocities of the bodies at P and Q.

By the same, by construction, are directly proportional to $AP : BQ$.

That is, proportional to the perpendiculars to point D hanging from the tangents.

And so, points S, D, T are in one straight line.

And by a similar argument, S, E, V are also in a straight line.

And therefore the centre is at the intersection of the straight lines TD, VE (suitably extended).

Q.E.D.

Proposition VI. Theorem V. *Let a body be revolving around a fixed point in some orbit in non-resisting space. If it would describe an arc in some small time, and the arrow of that arc would be found which bisects the chord and is produced to the centre of the force, then the centripetal force will be directly as the arrow, and twice the inverse of time.*[25] [26]

[22] Centre here is used in the sense of where centripetal force is acting to. It is not the centre of mass or of gravity or of any particular geometric significance. For example, if this was an ellipse, we would be finding one of the foci, rather than the geometric centre of the ellipse.

[23] We are given the velocities at (three) points on the figure. We assume that force acts towards a fixed centre. Find that centre.

[24] $PA : QB = V_Q : V_P$ and $QB : RC = V_R : V_Q$

[25] By being twice proportional means that it is proportional to the square: $A \propto B \times B$

[26] This proposition says that the centripetal force to a point S is proportional to the deflection of the arc away from the tangent, divided by the square of the time. $F \propto \frac{A}{t^2}$

Figure 5.3: Semicircle

For the arrow in given time is proportional to force (by cor. 4 of Prop. I).

Increasing time in any ratio, the arc would grow in the same ratio, and the arrow is increased in that ratio duplicated (by cor.2 & 3 of Lemma XI).

So the arrow is proportional to the force and twice proportional to the time.

Dividing away the duplication of the ratio of times, and force is proportional to the arrow directly and twice proportional to the inverse of time.[27]

Q.E.D.

Corollary I If body P revolving around a centre S describes a curved line APQ; a straight tangent ZPR would be made to that curve through P,

and to that tangent, from any other point Q of the curve, is drawn QR parallel to the line SP, and then QT is drawn perpendicular to that line SP, centripetal force will be proportional to the solid[28] $\frac{SP^2 \times QT^2}{QR}$.

Corollary 2 By the same argument, centripetal force is reciprocal to the solid $\frac{SY^2 \times QP^2}{QR}$ if the line SY to the centre of the force would be perpendicular to the produced tangent PR.

For, the rectangles $SY \times QP$ and $SP \times QT$ are equal.

Corollary 3

If the orbit either is a circle, or touches a circle with the same centre, or cuts a circle with the same centre, the same having the same radius of curvature as at the point P: centripetal force will be reciprocally proportional to the solid $SY^2 \times PV$. For $PV = \frac{QP^2}{QR}$.

Corollary 4

In the same setup[29], centripetal force is directly proportional to velocity twice, and that chord inversely, for velocity is reciprocal to the perpendicular SY by Prop. I. Cor. 1.

Corollary 5

And so if given a curved figure APQ, and in that figure, given a point S, to which centripetal force always pulls, to find a law of centripetal force, where a

[27] Divide both sides by t^2 and we get $\frac{A}{t^2} \propto F$, or rearranging $F \propto \frac{A}{t^2}$

[28] a quantity with dimension of length cubed, or volume.

[29] the setup of corollary 3

body P is always pulled from its straight path into the perimeter of that figure, describing a path around the figure: The force is reciprocally proportional to either the solid $\frac{SP^2 \times QT^2}{QR}$ or the solid $SY^2 \times PV$.

Examples are in the following problems.

Proposition VII. Problem II. *A body revolves in the circumference of a circle. A law*[30] *of the centripetal force tending to a given point is required.*[31]

[answer: $F \propto \frac{1}{SP^2 \times PV^3}$]

Figure 5.4: Body revolving in a circle

The circumference of the circle is $VQPA$. The given point, to which the force tends, is S. The body is in the line of the circumference at P. Near to it, is Q which can be moved. Line PRZ is tangent to the circle at P. From point S is drawn the chord PV. VA is a diameter of the circle, joined to AP. And to SP would be produced the perpendicular QT, which intersects the tangent PR at Z. And finally, line LR is drawn through point Q, itself parallel to SP, and meets both the circle at L and the tangent PZ at R.

And, by reason of the similarity of the triangles $\triangle ZQR$, $\triangle ZTP$, $\triangle VPA$: $((RP^2 = QRL) : QT^2) = AV^2 : PV^2$

And by the same idea, $\frac{QRL \times PV^2}{AV^2} = QT^2$.

This would be multiplied into $\frac{SP^2}{QR}$. The points P, Q coalescing would draw PV to RL. Thus, $\frac{SP^2 \times PV^3}{AV^2}$ becomes equal to $\frac{SP^2 \times QT^2}{QR}$.

[30] See 1.2 for discussion of whether this is *A law* or *The Law*. What is being asked for is a function from position to magnitude of the force. The direction of the force is always known (being towards the fixed point).

[31] So, we can pick any point arbitrarily (although I think it is assumed that this point is within the circle) and pretend that the centripetal force is acting towards that point. What is the force?

Thus, $\frac{SP^2 \times PV^3}{AV^2}$ becomes $\frac{SP^2 \times QT^2}{QR}$.

Therefore (by corollaries 1 and 5 of Proposition VI) centripetal force is reciprocal to $\frac{SP^2 \times PV^3}{AV^2}$. That is (for given AV^2, it is reciprocal to the square of the distance SP, and the cube of PV, combined. [32]

Corollary 1

And so, if the given point S, to which centripetal force always tends, would be located on the circumference of the circuit, and regarded as V; then centripetal force will be reciprocal to the "quadrato-cubis" [33] of the length SP. [34]

Corollary 2

Force, which the body P in circle $APTV$ around centre of force S would revolve, is to force, which the same body P in the same circle, and in the same period of time around another centre of force R revolving, is how $RP^2 \times SP$ to the cube of the line SG, which to the first centre of force S to tangent to the orbit PG is drawn, and of the distance of the body to second centre of force is parallel.

Figure 5.5: Circle

For, by contruction, the first force is to the last force how $RP^2 \times PT^3 : SP^2 \times PV^3$, that is how $SP \times RP^2 : \frac{SP^3 \times PV^3}{PT^3}$, or $: SG^3$ (by the similarity of the triangles $\triangle PSG$, $\triangle TPV$).

Proposition VIII. Problem III. *Let a body be moved in a semi-circle PQA.* [35] *To this, a law of centripetal force tending to some distant point S where every line drawn to it (for example PS, RS) is parallel,*[36] *is required.*

[answer $F \propto \frac{1}{PM^3}$]

To the centre C of the semicircle is drawn a semi-diameter cutting the parallels at M and N and joined to CP.

By reason of the similarity of triangles $\triangle CPM$, $\triangle PZT$, $\triangle RZQ$: $CP^2 : PM^2 = PR^2 : QT^2$.

[32] AV^2 is fixed because it is the diameter.

[33] the fifth power: $(SP^3) \times (SP^2) = SP^5$

[34] If the point S is on the circumference, we can regard S and V as the same point, and so $SP = PV$ and so $F \propto \frac{1}{SP^2 \times SP^3} = \frac{1}{SP^5}$

[35] P, Q, A are just points on the semicircle. P, A are not the endpoints.

[36] S is a point at infinity.

Figure 5.6: Semicircle

And, from the nature of circles, PR^2 equals the rectangle $QR \times (RN + QN)$[37]
Or, if points P,Q coaelesce(sp?), $PR^2 = QR \times 2PM$.
Therefore, $CP^2 : PM^2 = \frac{QR \times 2PM}{QT \times QT}$.
By this idea, $\frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$
And, $\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}$
Therefore, (by corollaries 1 & 5 of Prop.VI) centripetal force is reciprocal to $\frac{2PM^3 \times SP^2}{CP^2}$. That is (neglecting the determined[38] ratio $\frac{2SP^2}{CP^2}$), it is reciprocal to PM^3.
Q.E.I.

Proposition IX. Problem IV.. *If a body would be rotating in a spiral PQS, cutting all radiuses SP, SQ, and so on at a given angle, a law of the centripetal force tending to the centre of the spiral is required.*

[answer $F \propto \frac{1}{SP^3}$]

Figure 5.7: Part of a spiral with constant angle

Given an indefinite small angle $\angle PSQ$, every any creates a figure like $SPRQT$.

[37] This is typeset in the original text as: $QR \times \overline{RN + QN}$.
[38] fixed: SP^2 is fixed since S is at (or near) infinity, and CP^2 always fixed, by diameter of semicircle.

Therefore, if given ratio $\frac{QT}{QR}$[39], then $\frac{QT^2}{QR} \propto QT$, and then, $QT \propto SP$.

Now, if the angle would be changed, the straight line QR across the angle QPR will be proportional (by Lemma XI) to the square of PR or QT.

So $\frac{QT^2 \times SP^2}{QR} \propto SP^3$. By corollaries 1 and 5 of Proposition VI, centripetal force is reciprocal to the cube of the distance SP.

Lemma XII. *All parallelograms around a given ellipse or hyperbola, described by conjugate diameters, are equal to each other.*

Shown from conics.

Proposition X. *If a body is moved in an ellipse: A law of centripetal force tending to the centre of the ellipse is required.*

[answer $F \propto$ radius]

Figure 5.8: Body rotating in an ellipse

CA and CB are the semiaxes of the ellipse; GP, DK are another pair of conjugate diameters. PF, QT are perpendicular to those diameters. Qv is an ordinate[40] applied to the diameter GP.

Let parallelogram $QvPR$ be completed, and (from conics) $PvG : Qv^2 = PC^2 : CD^2$, and (from the similarity of triangles $\triangle QvT$, $\triangle PCF$) $Qv^2 : QT^2 = PC^2 : PF^2$.

Combining these ratios, $PvG : QT^2 = PC^2 : CD^2$ and $PC^2 : PF^2$ that is, $vG : \frac{QT^2}{Pv} = PC^2 : \frac{CD^2 \times PF^2}{PC^2}$.

Substitute QR for Pv and (by lemma XII) substitute $BC \times CA$ pro $CD \times PF$, and so substitute (points P & Q coming together) $2PC$ for vG,

[39] which is fixed because the angles are fixed.

[40] An ordinate is a line parallel to the tangent. In this case, the ordinate Qv is to be parallel to the tangent through P.

& so $\frac{QT^2 \times PC^2}{QR} = \frac{2BC^2 \times CA^2}{PC}$.

Therefore, centripetal force is (by corol. 5 prop. VI) reciprocal to $\frac{2BC^q \times CA^2}{PC}$; that is (because of fixed $2BC^2 \times CA^2$[41]) reciprocal to $\frac{1}{PC}$; that is, directly proportional to PC.

Q.E.I. [42]

[41] because CA and BC are the semiaxes of the ellipse

[42] This is quite a counter-intuitive result: If a body is orbiting an ellipse and the force is to the centre, then the centripetal force is proportional to the distance. If this were the case for gravity everywhere, the universe would be very strange indeed - the further away two bodies are, the greater their effect on each other.
Prop. X says that if we have centripetal force to the centre of the ellipse, the force is proportional to the distance. Prop. XI says if we have centripetal force to a focus of the ellipse, force is inversely proportional to the square of the distance. The two meet in the case where a focus is at the centre (and hence both focuses are at the centre). In this case, we have a circle and the radius is constant, so both propositions work. There arises the question (which Newton deals with later in the book), which is reality in the case of planets - Are the planets drawn to the centre of their orbit, or to a focus? I believe that the focuses of the orbits of the planet are very close to the centre, so it is a very delicate choice. But it is fairly obvious that if the former is the case, and this same force affects everything in the universe, then some tiny movement a long way away would cause massive disruption here.

Chapter 6

Section III, On the motion of bodies in eccentric conic sections

I present only the first result from Section III, a proposition and problem relating to the inverse square law. It is a substantial result and makes use of much of the work in the preceeding sections. It is quite similar in character to Proposition X, but Newton will later go on to say that Proposition XI is a model for gravity, rather than Proposition X.

I have reworked the layout and notation of the proof, and added in a lot of commentary, but the spirit of the proof is left intact.

Proposition XI. Problem VI. *If a body is revolving in an ellipse, a law of the centripetal force to a focus of the ellipse is required.*

Let S be a focus of the ellipse.[1]

Draw a secant SP intersecting the diameter DK at E and ordinate Qv at x.

Complete the parallelogram $QxPR$.[2]

Observe EP to be equal to the major semiaxis AC, because: a line HI, parallel to EC, acts to the other focus of the ellipse, H. [3]

since S and H are both focuses, they are the same distance from the centre C.

$ES = EI$ [4]

$EP = \frac{PI+PS}{2}$ (because: $HI \parallel PR$ [$HI \parallel EC \parallel PR$] and $\angle IPR = \angle HPZ$.) which is furthermore the [major] axis, $2AC$. [5]

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[1]Specifically, the focus to which the centripetal force is acting.
The setup for this proposition is implicitly inherited from Proposition X. There are, therefore, a lot of definitions carried forward from that proposition, including:
Let C be the centre of the ellipse.
Let P and Q be two points on the ellipse.
Let DK, GP be conjugate diameters.

[2]This defines point R

[3]Point I is defined here.

[4]The triangles ESC and ISH are similar. The base of the former is twice the base of the latter, and so all of the sides are in this relationship. So SI is twice SE, so $EI = SI - ES = EI$.

[5]since E is halfway between I and S, the distance to E is the mean of the distances for I

Figure 6.1: Body revolving in an ellipse, with force to a focus

To SP, drop the perpendicular QT.
Let $L = \frac{2BC^2}{AC}$, the principle straight line[6] of the ellipse.
$L \times QR : L \times Pv = QR : Pv$ [7]
$= PE : PC = AC : PC$ [8]
$L \times Pv : GvP = L : Gv$ [9]
$Gv \times vP : Qv^2 = PC^2 : CD^2$
[10]

If we let $Q \to P$, [11] then $Qv^2 \to Qx^2$.
$Qx^2 : QT^2 = Qv^2 : QT^2 = EP^2 : PF^2 = CA^2 : PF^2 = CD^2 : CB^2$ [12]

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and S, which is itself $PS + PH$ [i.e. $PI + PS = PS + PH$]
In fact, given any point on the ellipse, the sum of the distances to each focus is equal to the major axis, $2AC$

[6] latin: *latus rectum*. The latus rectum is the length of line through a focus of the ellipse, parallel to the minor axis and perpendicular to the major axis, between the points where this line crosses the ellipse.

[7] obvious, by cancellation of L

[8] Note that in the text, this relationship is expressed as (PE or AC) : PC
$QR = Px$ since they are opposite sides of the same parallelogram. The triangle Pxv which is similar to PEC. So the ratio of the various sides is the same on both triangles. So $QR : Pv = Px : Pv = PE : PC$. Recall $AC = PE$ so $PE : PC = AC : PC$.

[9] $L \times Pv : Gv \times vP = L : Gv$ by cancelling Pv

[10] See Useful Result 2. We are applying that result on the lines GvP, QvQ', DCK, PCG (where Q' is the point on the ellipse reached by extending Qv and $Qv = vQ'$) intersecting at points v and C. This gives the ratio: $Gv \times vP : Qv \times vQ' = PC \times CG : DC \times CK$ We then make substitutions: $vQ' = Qv$, $CG = PC$, $CK = DC$, and square both sides.

[11] Here, and for the rest of the proof, we are talking about the limiting case when Q converges to P.

[12] $\angle EFP = \angle QTX$ are right angles. $\angle TxQ = \angle IPR = \angle PEF$. So the triangles $\triangle EFP$ and $\triangle QTX$ are similar, and so the ratios of their sides are equal. $Qx : QT = PE : PF$ $Qx^2 : QT^2 = PE^2 : PF^2$ But, $EP = CA$ so $Qx^2 : QT^2 = CA^2 : PF^2$

And by combining all of these ratios:
$L \times QR : QT^2 = AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2$ [13]
$= 2CB^2 \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2$ [14]
$= 2PC : Gv$ [15]
As $Q \to P$ come together, then $2PC$ and Gv become equal. [16]
Therefore, $L \times QR = QT^2$ [17]
Multiplying this into $\frac{SP^2}{QR}$, we get: $L \times SP^2 = \frac{SP^2 \times QT^2}{QR}$
Therefore (by corollaries 1 & 5 of proposition 5), the centripetal force is reciprocal to $L \times SP^2$, that is, it is reciprocal to SP^2.
Q.E.I.

[13] $L \times QR : L \times PV = AC : PC$
$L \times PV : Gvp = L : Gv$
$GvP : Qv^2 = PC^2 : CD^2$
$Qv^2 : QT^2 = CD^2 : CB^2$
The terms on the left hand side cancel in a chain and we are left with $L \times QR : QT^2$. Multiply the RHS terms together. We are left with the desired result.

[14] For this second equality, recall that $L = \frac{2BC^2}{AC}$. So, $L \times AC = BC^2$. Substitute this in, and we get the second equality.

[15] This can be seen by easy cancellation.

[16] $v \to P$, so $Cv \to PC$, and GC = GP

[17] from the big combination on 55/20

Appendix A

Useful Results

The results in this section are not presented in Newton's text. However, the reader seems to be expected to know them from works such as Euclid's *Elements* and from geometry in general.

Since they were not all immediately obvious to me, especially when used in the middle of a proof, I have included them here.

Useful Result 1 *Two triangles, with the same base and the same height (measured perpendicular to the base), have the same area.*

Useful Result 2 *If we have an ellipse, with two pairs of parallel chords, $AXC \parallel A'X'C'$ and $DXF \parallel D'X'F'$, the chords intersecting at X and X', then the following equality holds:*

$$AX \times XC : DX \times XF = A'X' \times X'C' : D'X' \times X'F'$$

Euclid III.35 provides the proof for this in the specicial case of a circle, where the above ratio is always 1:1 no matter which points are chosen.

It is not entirely clear how Newton intended to prove the general elliptical case, but it can be seen using projective techniques by tilting the circle so that it looks like the desired ellipse.

Useful Result 3 (Componendo) *If*

$$\frac{a}{b} = \frac{c}{d}$$

then

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Bibliography

[1] Carl B. Boyer and Uta C. Merzbach. *A History Of Mathematics.* John Wiley & sons, 1991.

[2] B.J. Hayes and W.F. Mason. *The Tutorial Latin Grammar.* University Tutorial Press, 1924.

[3] Alexandre Koyré and I. Bernard Cohen, editors. *Isaac Newton's Philosophiae Naturalis Principia Mathematica, with variant readings.* Cambridge University Press, 1972.

[4] James Morwood, editor. *The Pocket Oxford Latin Dictionary.* Oxford University Press, second edition, 1994.

[5] Isaac Newton. *Philosophiae Naturalis Principia Mathematica.* Third edition, 1726. Facsimile reproduction in [3].